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Topics on Kac-Moody Lie algebras

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In this note we introduce Kac-Moody Lie algebras, their representation theory, and relation with theory of differential equations.

I. Kac-Moody Lie algebras ([6],[I4],[I5])

Let $C=(c_{ij})$ be a n×n-matix satisfying following conditions;

$c_{ij}\in\mathbb{Z}$, $c_{ii}=2$, $c_{ij}\leq 0$ $(i\neq j)$, $c_{ij}=0 \Leftrightarrow c_{ji}=0$,

and define Lie algebra $\mathfrak{L}(C)$ over $\mathbb{C}$ with generators $\{h_i, e_i, f_i \ i=I,\cdots,n\}$ and relations;

$$\begin{cases} [h_i,h_j]=0 \ , \ [h_i,e_j]=c_{ij}e_j \ , \ [h_i,f_j]=-c_{ij}f_j \ , \ [e_i,f_j]=\delta_{ij}h_i \ , \\ (\mathrm{ad}\ e_i)^{-c_{ij}+I}e_j=0 \ , \ (\mathrm{ad}\ f_i)^{-c_{ij}+I}f_j=0 \ . \end{cases}$$

$\mathfrak{L}=\mathfrak{L}(C)$ is called Kac-Moody Lie algebra.

$\mathfrak{L}$ has $\Gamma=\mathbb{Z}^n$ grading with deg $h_i=(0,\cdots,0)$, deg $e_i=(0,\cdots,\overset{i}{I},\cdots,0)=a_i$,and deg $f_i=-a_i$.

Put $\mathfrak{L}_a=\{x\in\mathfrak{L} \mid \deg x=a, (a\in\Gamma)\}$, $m_a=\dim\mathfrak{L}_a$, $\Delta=\{a\in\Gamma\setminus\{0\} \mid m_a\neq 0\}$.

From defining relations $\mathfrak{L}$ has vector space decomposition

$\mathfrak{L}=\mathfrak{h}\oplus\mathfrak{L}_+\oplus\mathfrak{L}_-$ where $\mathfrak{h}=\sum_i\mathbb{C}h_i$, $\mathfrak{L}_+=\langle e_I,\cdots,e_n\rangle$, $\mathfrak{L}_-=\langle f_I,\cdots,f_n\rangle$.

So Δ is disjoint union of $\Delta_+=\Gamma_+\cap\Delta$ and $\Delta_-=(-\Gamma_+)\cap\Delta$

$(\Gamma_+=\mathbb{Z}_+a_I+\cdots+\mathbb{Z}_+a_n)$

For each i ,define s_i GL(Γ) by

$s_i(a_j)=a_j-c_{ij}a_i$

and $W=\langle s_i \mid i=I,\cdots,n\rangle$.

If C is decomposable,i.e. has a permutation matrix P such that

$PCP^{-I}=\left(\begin{array}{c|c} C_I & O \\ \hline O & C_2 \end{array}\right)$, then $\mathcal{L}(C)\cong\mathcal{L}(C_I)+\mathcal{L}(C_2)$, $W(C)\cong W(C_I) \times W(C_2)$.

So we assume C is indecomposable.

A finite-dimensional complex simple Lie algebra is isomorphic to a $\mathcal{L}(C)$ whose W(C) is finite group.Equivalent condition on C to finiteness of W(C) is that there exist positive numbers $d_I,\cdots,d_n$ such that $\begin{pmatrix} d_I & & \\ & \ddots & \\ & & d_n \end{pmatrix}\cdot C$ is a positive definite symmetric matrix.

If we replace "positive definite" by "positive semi-definite" in this condition, we get a new class of infinite-dimensional Lie algebras. They are called Euclidean (or affine) Lie algebras.We deal with Euclidean case hereafter.

2.Character formula and denominator formula

Put $V=\mathbb{C}\otimes_{\mathbb{Z}}\Gamma$,and for $\lambda\in\mathfrak{h}^*$ let $c_\lambda:W\rightarrow V$ be the I-cocycle,i.e.satisfies

$c_\lambda(w_Iw_2)= w_Ic_\lambda(w_2) + c_\lambda(w_I)$,such that $c_\lambda(s_i)=\lambda(h_i)a_i$ $(i=I,\cdots,n)$.

If $\lambda(h_i)\in\mathbb{Z}$ for all i, λ is said integral,and moreover if $\lambda(h_i)\geqq 0$, λ is said dominant integral. It is easy to see

$$\begin{cases} \lambda\text{:integral} \Rightarrow c_\lambda(W)\subset\Gamma \\ \lambda\text{:dominat integral} \Rightarrow c_\lambda(W)\subset\Gamma_+ \end{cases}$$

To each dominant integral,there exists $\mathcal{L}$ -module V^λ,unique up to isomorphism ,with the following property;

$$0\neq\exists v\in V \text{ s.t. } \begin{cases} e_iv=0 \;,\; h_iv=\lambda(h_i)v \;,\; f_i^{\lambda(h_i)}v=0 \quad (i=I,\cdots,n) \\ V=\sum\mathbb{C}f_{i_I}f_{i_2}\cdots f_{i_k}v \end{cases}$$

V^λ is called standard module with highest weight λ.

For $a\in\Gamma_+$,let $V^\lambda_a=\sum_{a_{i_I}+\cdots a_{i_k}=a}\mathbb{C}f_{i_I}\cdots f_{i_k}v$, and put

$ch\,V^\lambda=\sum_{a\in\Gamma_+}\dim V^\lambda_a e(-a)$ (chV^λ is in $\mathbb{Z}[[e(-a_I),\cdots,e(-a_n)]]$.)

Let $\rho \in \mathfrak{h}^*$ be defined by $\rho(h_i)=I$ for all i.We have

character formula ([5],[8],[IO])

$$\mathrm{ch}\, V^{\lambda}=\frac{\sum_{w\in W}\det(w)e(-c_{\lambda+\rho}(w))}{\sum_{w\in W}\det(w)e(-c_{\rho}(w))}$$

and

denominator formula

$$\sum_{w\in W}\det(w)e(-c\quad(w)) = \prod_{a\in\Delta_+}(I-e(-a))^{m_a}$$

If we make specialization $e(-a_i)\mapsto q^{m_i}$ $(m_I,\cdots,m_n \geqq 0)$ to both sides of denominator (or character) formula,we have various combinatorial identities. ([7],[8],[I2],[I6])

3.K-dV equation and construction of representation

Let $u(x),V_{n\ell}(x)$ $(n=I,2,3,\cdots\ ,I\leq\ell\leq 2n-2)$ be functions in infinite many variables $x=(x_I,x_3,x_5,\cdots)$.From the compatibility conditions of linear partial differential equations

$$\left.\begin{aligned}&\left\{\left(\frac{\partial}{\partial x_1}\right)^2+u(x)\right\}\psi(x)=\lambda\psi(x)\\&\frac{\partial\psi}{\partial x_{2n-1}}=\left\{\left(\frac{\partial}{\partial x_1}\right)^{2n-1}+V_{nI}\left(\frac{\partial}{\partial x_1}\right)^{2n-3}+\cdots+V_{n\ 2n-2}\right\}\psi\quad(n=I,2,\cdots)\end{aligned}\right\}$$

we obtain non-linear differential equations on u,and $V_{n\ell}$ are expressed by differential polynomials of u.These differential equations are called K-dV equations. A function $\tau(x)$ is called τ-function,if $u=\left(\frac{\partial}{\partial x_1}\right)^2\log\tau$ is a solution of K-dV equations.The space of infinitesimal trasformations of τ-functios forms a Lie algebra $\mathcal{L}$ of linear trans-formations on $C[x_I,x_3,\cdots]$. Date,Jimbo,Kashiwara and Miwa find ([I]) that $\mathcal{L}\cong\mathcal{L}\begin{pmatrix}2&-2\\-2&2\end{pmatrix}$ and $\mathcal{L}$-module $C[x_I,x_3,\cdots]$ coincides with the standard module constructed by Lepowsky and Wilson.([II]). Kac,Kazhdan,Lepowsky and Wilson construct standard modules for other Euclidean Lie algebras in analogous way toL-W,and D-J-K-M show they correspond to some non-linear equations like K-dV.([2],[3] ,[9])

4. Remarks

(I) About classification and realizations of Euclidean Lie algebras,c.f. [6],[I5].

(2) Let $\mathfrak{R}\subset\mathcal{L}(C)$ be the maximam homogenious ideal with $\mathfrak{R}\cap\mathfrak{f}=0$ .Strictly speaking, $\mathfrak{g}(C)=\mathcal{L}(C)/\mathfrak{R}$ is Kac-Moody Lie algebra.But finite or Euclidean case $\mathfrak{R}=0$,and we conjecture that $\mathfrak{R}=0$ for any case.

(3) Character formula and denominator formula are analogy of finite case, and proved for $\mathfrak{g}(C)$ with symmetrizable C,i.e. there exist positive numbers $d_I, \cdots, d_n$ such that $\begin{pmatrix} d_I & & \\ & \ddots & \\ & & d_n \end{pmatrix}C$ is symmetric matrix.

(4) Original K-dV equation is obtained by to put $x_5, x_7, \cdots$ constant.

(5) Frenkel and Kac ([4]) construct standard modules in different way from K-K-L-W.It will be interesting to describe isomorphism explicitly between K-K-L-W's module and F-K's.

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